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THE NON-LINEAR DYNAMICS OF ELASTIC RODS*

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The general equations of non-linear dynamics of elastic rods are examined taking tension, transverse shear, eccentricity, rotational inertia, and also initial stresses into account. A second-order theory is constructed for Timoshenko and classical-type models. A variational formulation is given for the linearized problem. Tension and shear effects are examined in the problem of the stability of a compressed column.

1. Geometry and kinematics. A rod is considered below to be a deformable material line whose particles are solids /1/. A Lagrange coordinate s, $0 \leqslant s \leqslant l$ is introduced. This usually an arc coordinate in a reference configuration. The rod motion is determined by the time dependence of the radius-vector $\mathbf{r}(s, t)$ and the rotation tensor $\mathbf{P}(s, t)$ for each particle. Internal interactions are given by the force vector $\mathbf{Q}(s, t)$ and moment vector $\mathbf{M}(s, t)$ with which a particle with coordinate $s+0$ acts on a neighbour $s-0$ ($\mathbf{Q}$ and $\mathbf{M}$ change when the reference direction s is reversed).

To assign an angular orientation, an orthogonal triple $\mathbf{e}_k$ is associated with each particle according to a certain rule; it is often assumed, say, that $\mathbf{e}_{30} = \mathbf{r}_0'$ ($(\ldots)' = \partial/\partial s$; the zero subscript marks quantities in the reference configuration). By the definition of the rotation tensor $\mathbf{e}_k = \mathbf{P}\cdot\mathbf{e}_{k0}$, $\mathbf{P} = \mathbf{e}_k\mathbf{e}_{k0}$. Here and henceforth, the language of the direct tensor calculus is used /2/. The curvature vector and rod twist are introduced by the relationships $\mathbf{e}_k' = \mathbf{\Omega} \times \mathbf{e}_k$, $\mathbf{\Omega} = {}^1/_2\mathbf{e}_k \times \mathbf{e}_k'$ As will be shown below, the vectors

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$$\Gamma = \mathbf{r}' - \mathbf{P}\cdot\mathbf{r}_0', \quad \varkappa = \Omega - \mathbf{P}\cdot\Omega_0 \tag{1.1}$$

are measures of the deformation.

The equality $\mathbf{P}' = \varkappa \times \mathbf{P}$ holds; it shows that $\varkappa$ is defined only by the non-uniformity of the distribution of the rotations and is independent of the method of assigning $\mathbf{e}_k$.

In the classic Kirchhoff-Clebsch model $\Gamma = 0$ is assumed; this is interpreted as the absence of tension and transverse shear. The non-linear dynamics for this case was examined in /3/.

The velocity vector $\mathbf{v}(s, t)$ and angular velocity vector $\omega(s, t)$ are introduced for each particle

$$\mathbf{r}^{\cdot} = \mathbf{v}, \quad \mathbf{P}^{\cdot} = \omega \times \mathbf{P} \quad ((\ldots)^{\cdot} = \partial/\partial t)$$

From the equalities $(\mathbf{r}^{\cdot})' = (\mathbf{r}')^{\cdot}$, $(\mathbf{P}^{\cdot})' = (\mathbf{P}')^{\cdot}$ it follows that

$$\mathbf{v}' - \omega \times \mathbf{r}' = \Gamma^{\cdot} - \omega \times \Gamma, \quad \omega' = \varkappa^{\cdot} - \omega \times \varkappa$$

The translation vector $\delta\mathbf{r}$ and a small rotation vector $\delta\mathbf{o}$ are given during the variation of the actual configuration for each particle. Unlike $\delta\mathbf{r}$, $\delta\mathbf{o}$ is not a variation of a vector but only denotes the accompanying vector for an antisymmetric tensor $\delta\mathbf{P}\cdot\mathbf{P}^T = \delta\mathbf{o} \times \mathbf{E}$ $((\ldots)^T$ is the transposition symbol, and $\mathbf{E}$ is the unit tensor) so that $\delta\mathbf{P} = \delta\mathbf{o} \times \mathbf{P}$. We obtain for the variations

$$\delta\mathbf{r}' - \delta\mathbf{o} \times \mathbf{r}' = \delta\Gamma - \delta\mathbf{o} \times \Gamma = \mathbf{e}_k\delta\Gamma_k \tag{1.2}$$

$$\delta\mathbf{o}' = \delta\varkappa - \delta\mathbf{o} \times \omega = \mathbf{e}_k\delta\varkappa_k$$

$(\Gamma_k = \Gamma\cdot\mathbf{e}_k$ are components in the basis $\mathbf{e}_k)$.

An expression for the virtual work for a solid is also needed below. The radius-vector of an arbitrary point of a body is $\mathbf{R} = \mathbf{r} + \mathbf{x}$, where $\mathbf{r}$ is the radius-vector of a pole. Here

$$\mathbf{R}^{\cdot} = \mathbf{r}^{\cdot} + \omega \times \mathbf{x}, \quad \mathbf{R}^{\cdot\cdot} = \mathbf{r}^{\cdot\cdot} + \omega^{\cdot} \times \mathbf{x} + \omega \times (\omega \times \mathbf{x}), \quad \delta\mathbf{R} = \delta\mathbf{r} + \delta\mathbf{o} \times \mathbf{x}$$

The work of the inertia force equals

$$-\int \mathbf{R}^{\cdot\cdot}\cdot\delta\mathbf{R}\,dm = -m\,[(\mathbf{v} + \omega \times \varepsilon)^{\cdot}\cdot\delta\mathbf{r} + (\varepsilon \times \mathbf{v}^{\cdot} + (\mathbf{J}\cdot\omega)^{\cdot})\cdot\delta\mathbf{o}] \tag{1.3}$$

$$m\varepsilon = \int \mathbf{x}\,dm, \quad m\mathbf{J} = \int (x^2\mathbf{E} - \mathbf{x}\mathbf{x})\,dm$$

where ε and $\mathbf{J}$ are the eccentricity vector and the inertia tensor, and m is the body mass.

The inertial properties of the rod are given by the functions $\rho(s)$, $\varepsilon(s, t)$, $\mathbf{J}(s, t)$, where

$$\varepsilon^{\cdot} = \omega \times \varepsilon, \; \mathbf{J}^{\cdot} = \omega \times \mathbf{J} - \mathbf{J} \times \omega \tag{1.4}$$

We set $m = \rho ds$ in (1.3) when calculating the work of the inertia force for the element ds.

2. Fundamental variational equation and its corollary. Underlying the mechanics of elastic rods is the D'Alembert-Lagrange variational equation /2, 4/. For the section of the rod $s_1 \leqslant s \leqslant s_2$ we postulate it in the form

$$\int_{s_1}^{s_2} \{[\mathbf{q} - \rho(\mathbf{v}^{\cdot} + \varepsilon^{\cdot\cdot})]\cdot\delta\mathbf{r} + [\mathbf{m} - \rho(\varepsilon\times\mathbf{v}^{\cdot} + (\mathbf{J}\cdot\omega)^{\cdot})]\cdot\delta\mathbf{o} - \delta\Pi\}\,ds + (\mathbf{Q}\cdot\delta\mathbf{r} + \mathbf{M}\cdot\delta\mathbf{o})\big|_{s_1}^{s_2} = 0 \tag{2.1}$$

where the external force $\mathbf{q}$ and moment $\mathbf{m}$ as well as the strain energy Π referred to unit length are introduced. Since (2.1) is true for an arbitrary interval, we have

$$[\mathbf{Q}' + \mathbf{q} - \rho\,(\mathbf{v}^{\cdot} + \varepsilon^{\cdot\cdot})]\cdot\delta\mathbf{r} + [\mathbf{M}' + \mathbf{m} - \rho\,(\varepsilon \times \mathbf{v}^{\cdot} + (\mathbf{J}\cdot\omega)^{\cdot})]\cdot\delta\mathbf{o} + \mathbf{Q}\cdot\delta\mathbf{r}' + \mathbf{M}\cdot\delta\mathbf{o}' = \delta\Pi \tag{2.2}$$

The expression for Π is still unknown but it can be asserted that in "rigid" displacements, i.e., for $\delta\mathbf{r} = \mathrm{const}$, $\delta\mathbf{o} = 0$ and $\delta\mathbf{r} = \delta\mathbf{o} \times \mathbf{r}$, $\delta\mathbf{o} = \mathrm{const}$, we will have $\delta\Pi = 0$. We arrive at the equations

$$\mathbf{Q}' + \mathbf{q} = \rho\,(\mathbf{v}^{\cdot} + \varepsilon^{\cdot\cdot}) \tag{2.3}$$

$$\mathbf{M}' + \mathbf{r}' \times \mathbf{Q} + \mathbf{m} = \rho\,(\varepsilon \times \mathbf{v}^{\cdot} + (\mathbf{J}\cdot\omega)^{\cdot})$$

expressing the momentum balance and the moment of the impulse.

Taking account of (2.3) as well as (1.2), we reduce (2.2) to the form

$$\delta\Pi = \mathbf{M}\cdot\mathbf{e}_k\delta\varkappa_k + \mathbf{Q}\cdot\mathbf{e}_k\delta\Gamma_k \tag{2.4}$$

from which it follows that Π is a function of $\varkappa_k$ and Γ_k, where

$$\mathbf{M} = \mathbf{e}_k \partial \Pi / \partial \varkappa_k, \quad \mathbf{Q} = \mathbf{e}_k \partial \Pi / \partial \Gamma_k$$

For small strains it is possible to confine ourselves to the representation

$$\Pi = Q_{k0}\Gamma_k + \mathbf{M}_{k0}\varkappa_k + {}^1/_2\,(a_{kn}\varkappa_k\varkappa_n + b_{kn}\Gamma_k\Gamma_n + 2c_{kn}\varkappa_k\Gamma_n) \tag{2.5}$$

and then

$$\mathbf{M} = \mathbf{P}\cdot\mathbf{M}_0 + \mathbf{a}\cdot\boldsymbol{\varkappa} + \mathbf{c}\cdot\boldsymbol{\Gamma}, \quad \mathbf{Q} = \mathbf{P}\cdot\mathbf{Q}_0 + \mathbf{b}\cdot\boldsymbol{\Gamma} + \boldsymbol{\varkappa}\cdot\mathbf{c} \tag{2.6}$$

where $\mathbf{Q}_0$ and $\mathbf{M}_0$ and force and moment vectors in the reference configuration ($\mathbf{Q}_0 = Q_{k0}\mathbf{e}_{k0}$), while $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are stiffness tensors in the form $\mathbf{a} = a_{kn}\mathbf{e}_k\mathbf{e}_n$, etc. The tensors $\mathbf{a}$ and $\mathbf{b}$ are symmetric; $\mathbf{c}$ is not.

Together with the kinematic formulas from Sect.1, (2.3) and (2.6) form a complete system in the non-linear dynamics of elastic rod when tension and shear, eccentricity and rotational inertia as well as the initial stresses are taken into account. To go over to the classical model it is possible to set $\boldsymbol{\Gamma} = 0$ and get rid of the equation for $\mathbf{Q}$ in (2.6).

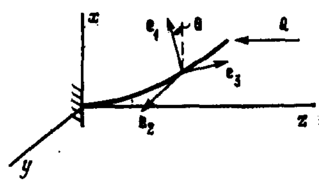
Within the framework of the elucidated "direct approach" /1/ it is impossible to determine the tensor elastic moduli $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Analysis of the appropriate three-dimensional problems (like the Saint-Venant minimum problem) is necessary to evaluate them /5, 6/. This also refers to the inertial characteristics of the rod.

As an illustration, we examine a straight rod clamped at the end $s = 0$ and compressed by a dead force Q at the end $s = l$ (sketch). We take an unloaded configuration as reference. Let us take the simplest modification of the stiffness tensors

$$\mathbf{a} = \Sigma a_k\mathbf{e}_k\mathbf{e}_k, \quad \mathbf{b} = \Sigma b_k\mathbf{e}_k\mathbf{e}_k, \quad \mathbf{c} = 0$$

where the directions $\mathbf{e}_k$ agree in the reference configuration with the directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the Cartesian x, y, z system. We assume that the strain occurs in the xz plane; then the rotation is given by one angle θ. In the equilibrium position $\mathbf{Q} = -Q\mathbf{k}$, $\mathbf{M}' = \mathbf{Q} \times \mathbf{r}'$.

Furthermore, using the geometric formulas and elasticity relationships



$$\mathbf{e}_1 = \mathbf{i}\cos\theta - \mathbf{k}\sin\theta, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{i}\sin\theta + \mathbf{k}\cos\theta$$

$$\boldsymbol{\varkappa} = \theta'\mathbf{j} = \mathbf{a}^{-1}\cdot\mathbf{M}, \quad \boldsymbol{\Gamma} = \mathbf{r}' - \mathbf{e}_3 = \mathbf{b}^{-1}\cdot\mathbf{Q}$$

we arrive at the equation

$$a_2\theta'' + Q\sin\theta + {}^1/_2 Q^2 (b_1^{-1} - b_3^{-1})\sin 2\theta = 0$$

The boundary conditions are $\theta(0) = \theta'(l) = 0$. After linearization, by solving the eigenvalue problem we arrive at an equation for the critical load

$$Q + Q^2/F = E; \quad E = \frac{\pi^2 a_2}{4l^2}, \quad F = \frac{b_1 b_3}{b_3 - b_1} \tag{2.7}$$

where E is the "Euler" critical load. If $F > 0$, i.e., the tensile stiffness b_3 is greater than the shear stiffness b_1, then (2.7) also has a negative root together with the usual positive root, which corresponds to buckling under tension. If $b_3 < b_1$, then instability is possible only under compression, and just in the case $4E \leqslant |F|$. These extraordinary effects are known for simplified models /7/.

3. Imposition of a small strain on a finite strain. Let $\mathbf{r}, \mathbf{P}, \mathbf{q}, \mathbf{Q}$ etc. receive small increments of the same order: $\mathbf{r}_1 = \mathbf{u}$, $\mathbf{P}_1 = \boldsymbol{\theta} \times \mathbf{P}$, $\mathbf{q}_1, \mathbf{Q}_1$, etc. Varying the general non-linear equations we obtain

$$\begin{aligned}
&\mathbf{Q}_1' + \mathbf{q}_1 = \rho\,(\mathbf{u} + \boldsymbol{\theta} \times \boldsymbol{\varepsilon})^{\cdot\cdot} \\
&\mathbf{M}_1' + \mathbf{u}' \times \mathbf{Q} + \mathbf{r}' \times \mathbf{Q}_1 + \mathbf{m}_1 = \rho\,[(\boldsymbol{\theta} \times \boldsymbol{\varepsilon}) \times \mathbf{v}^{\cdot} + \boldsymbol{\varepsilon} \times \\
&\quad \mathbf{u}^{\cdot\cdot} + (\mathbf{J}\cdot\boldsymbol{\theta}^{\cdot} + \boldsymbol{\theta} \times \mathbf{J}\cdot\boldsymbol{\omega})^{\cdot}] \\
&\mathbf{M}_1 = \boldsymbol{\theta} \times \mathbf{M} + \mathbf{a}\cdot\boldsymbol{\theta}' + \mathbf{c}\cdot\boldsymbol{\gamma} \quad (\boldsymbol{\gamma} = \mathbf{u}' - \boldsymbol{\theta} \times \mathbf{r}') \\
&\mathbf{Q}_1 = \boldsymbol{\theta} \times \mathbf{Q} + \mathbf{b}\cdot\boldsymbol{\gamma} + \boldsymbol{\theta}'\cdot\mathbf{c}
\end{aligned} \tag{3.1}$$

For instance, the last formula is derived thus: if follows from (2.6) that

$$\mathbf{Q}_1 = \mathbf{P}_1\cdot\mathbf{Q}_0 + \mathbf{b}_1\cdot\boldsymbol{\Gamma} + \mathbf{b}\cdot\boldsymbol{\Gamma}_1 + \boldsymbol{\varkappa}_1\cdot\mathbf{c} + \boldsymbol{\varkappa}\cdot\mathbf{c}_1$$

Using relationships of the type

$$\mathbf{b}_1 = \boldsymbol{\theta} \times \mathbf{b} - \mathbf{b} \times \boldsymbol{\theta}, \quad \boldsymbol{\Gamma}_1 = \boldsymbol{\gamma} + \boldsymbol{\theta} \times \boldsymbol{\Gamma}, \quad \boldsymbol{\varkappa}_1 = \boldsymbol{\theta}' + \boldsymbol{\theta} \times \boldsymbol{\varkappa}$$

we arrive at the expression for $\mathbf{Q}_1$ from (3.1).

The equalities

$$\boldsymbol{\omega}_1 = \boldsymbol{\theta}^{\cdot} + \boldsymbol{\theta} \times \boldsymbol{\omega}, \quad (\mathbf{J}\cdot\boldsymbol{\omega})_1 = \mathbf{J}\cdot\boldsymbol{\theta}^{\cdot} + \boldsymbol{\theta} \times \mathbf{J}\cdot\boldsymbol{\omega}$$

were used in varying the moment balance equation.

Eqs.(3.1) are equations of the first approximation in the theory of elastic rod stability. They are linear; the dependence of their coefficients on s and t is determined by the undisturbed motion. For a small deformation from the rest state of no stress, the quantities $\mathbf{Q}, \mathbf{M}$, $\mathbf{v}$ and $\boldsymbol{\omega}$ will be zero and (3.1) are converted into equations of the linear theory of rods.

4. Second-order effects. Considering the unknowns to be small quantities of the same order $\eta\,(\eta \to 0)$, we retain terms of order η^2 in all the equations. We use the following general representation of the rotation tensor /2/:

$$\mathbf{P} = \mathbf{E} + \mathbf{e}^* \times \mathbf{E} \sin\theta + \mathbf{e}^* \times \mathbf{E} \times \mathbf{e}^* (1 - \cos\theta) \tag{4.1}$$

where θ is the angle of rotation and $\mathbf{e}^*$ is the direction of its axis. Considering $\theta = O(\eta)$ and introducing the vector $\boldsymbol{\theta} = \theta\mathbf{e}^*$, we write (4.1) in the form

$$\mathbf{P} = \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E} + {}^1/_2\boldsymbol{\theta} \times \mathbf{E} \times \boldsymbol{\theta} + O(\eta^3) \tag{4.2}$$

Taking second-order terms into account, the geometric and kinematic relations will be

$$\begin{aligned} &\mathbf{e}_k = \mathbf{e}_{k0} + \boldsymbol{\theta} \times \mathbf{e}_{k0} + {}^1/_2\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \mathbf{e}_{k0}) + \ldots, \quad \boldsymbol{\varkappa} = \boldsymbol{\theta}' + \\ &\quad {}^1/_2\boldsymbol{\theta} \times \boldsymbol{\theta}' + \ldots \\ &\boldsymbol{\Gamma} = \boldsymbol{\gamma} - {}^1/_2\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \mathbf{r}_0') + \ldots \quad (\boldsymbol{\gamma} = \mathbf{u}' - \boldsymbol{\theta} \times r_0') \\ &\boldsymbol{\omega} = \boldsymbol{\theta}^{\cdot} + {}^1/_2\boldsymbol{\theta} \times \boldsymbol{\theta}^{\cdot} + \ldots, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \boldsymbol{\theta} \times \boldsymbol{\varepsilon}_0 + {}^1/_2\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \\ &\quad \boldsymbol{\varepsilon}_0) + \ldots \\ &\mathbf{J}\cdot\boldsymbol{\omega} = (\mathbf{J}_0 + \boldsymbol{\theta} \times \mathbf{J}_0 - {}^1/_2\mathbf{J}_0 \times \boldsymbol{\theta})\cdot\boldsymbol{\theta}^{\cdot} + \ldots \quad (\mathbf{J}_0 = J_{kn}\mathbf{e}_k\mathbf{e}_n) \end{aligned} \tag{4.3}$$

The equations of the impulse balance and the impulse moment and the elasticity relationships take the form

$$\begin{aligned} &\mathbf{Q}' + \mathbf{q} = \rho\,[\mathbf{u}^{\cdot\cdot} + \boldsymbol{\theta}^{\cdot\cdot} \times \boldsymbol{\varepsilon}_0 + {}^1/_2\,(\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\varepsilon}_0))^{\cdot\cdot}] \\ &\mathbf{M}' + (\mathbf{r}_0' + \mathbf{u}') \times \mathbf{Q} + \mathbf{m} = \rho\,[(\boldsymbol{\varepsilon}_0 + \boldsymbol{\theta} \times \boldsymbol{\varepsilon}_0) \times \mathbf{u}^{\cdot\cdot} + (\mathbf{J}_0 - \\ &\quad {}^1/_2\mathbf{J}_0 \times \boldsymbol{\theta})\cdot\boldsymbol{\theta}^{\cdot\cdot} + (\boldsymbol{\theta} \times \mathbf{J}_0\cdot\boldsymbol{\theta}^{\cdot})^{\cdot}] \\ &\mathbf{M} = \mathbf{M}_0 + \mathbf{a}_0\cdot\boldsymbol{\theta}' + \mathbf{c}_0\cdot\boldsymbol{\gamma} + \boldsymbol{\theta} \times (\mathbf{M}_0 + \mathbf{a}_0\cdot\boldsymbol{\theta}' + \mathbf{c}_0\cdot\boldsymbol{\gamma} + {}^1/_2\boldsymbol{\theta} \times \\ &\quad \mathbf{M}_0) + [{}^1/_2\mathbf{a}_0 \times \boldsymbol{\theta}' + \mathbf{c}_0 \times (\mathbf{u}' - {}^1/_2\boldsymbol{\theta} \times \mathbf{r}_0')]\cdot\boldsymbol{\theta} \\ &\mathbf{Q} = \mathbf{Q}_0 + \mathbf{b}_0\cdot\boldsymbol{\gamma} + \boldsymbol{\theta}'\cdot\mathbf{c}_0 + \boldsymbol{\theta} \times (\mathbf{Q}_0 + \mathbf{b}_0\cdot\boldsymbol{\gamma} + \boldsymbol{\theta}'\cdot\mathbf{c}_0 + {}^1/_2\boldsymbol{\theta} \times \\ &\quad \mathbf{Q}_0) - \boldsymbol{\theta}\cdot[(\mathbf{u}' - {}^1/_2\boldsymbol{\theta} \times \mathbf{r}_0') \times \mathbf{b}_0 + {}^1/_2\boldsymbol{\theta}' \times \mathbf{c}_0] \end{aligned} \tag{4.4}$$

Unlike the exact equations (Sect.2), all the elastic and inertial characteristics are considered known here: in place of the unknown "rotated" tensor $\mathbf{a} = a_{kn}\mathbf{e}_k\mathbf{e}_n$ there is $\mathbf{a}_0 = a_{kn}\mathbf{e}_{k0}\mathbf{e}_{n0}$, etc.

The second-order equations in the example from Sect.2 are

$$\begin{aligned} &Q_z = b_3\,(u_z' + \theta u_x' - {}^1/_2\theta^2) - b_1\theta\,(u_x' - \theta) = -Q \\ &Q_x = b_3\theta u_z' + b_1\,(u_x' - \theta - \theta u_z') = 0 \\ &M_y' + (1 + u_z')\,Q_x - u_x'Q_z = 0 \end{aligned} \tag{4.5}$$

For the **rectilinear form** $u_x = \theta = 0,\ b_3u_z' = -Q$. Linearizing (4.5) in the neighbourhood of this state, we **arrive** at the same result as in Sect.2. However, the exact moment balance equation is here used in (4.5), that contains third-order terms in the expression $\mathbf{r}' \times \mathbf{Q}$. Discarding these small terms, we obtain after **linearization**

$$a_2\theta'' + [Q + Q^2b_1^{-1}\,(1 - b_1b_3^{-1})^2]\,\theta = 0$$

which differs from the "exact" equation. However, the difference vanishes for $Q \ll b_3$. In this case $u_z' \ll 1$ and the non-linearities will be small, as is indeed assumed in the second-order theory.

5. The model with tension without transverse shear. This modification is hardly fundamental for applications. In this case $\Gamma_\alpha = 0\ (\alpha = 1,2)$. Eq.(2.4) takes the form

$$\delta\Pi = \mathbf{M}\cdot\mathbf{e}_k\delta\varkappa_k + Q\delta\Gamma \quad (Q = \mathbf{Q}\cdot\mathbf{e}_3,\ \Gamma = \Gamma_3)$$

and we obtain in the case of the quadratic approximation Π

$$\mathbf{M} = \mathbf{P}\cdot\mathbf{M}_0 + \mathbf{a}\cdot\boldsymbol{\varkappa} + \mathbf{c}\Gamma, \quad Q = Q_0 + b\Gamma + \mathbf{c}\cdot\boldsymbol{\varkappa} \tag{5.1}$$

Here the elastic properties are given by the tensor $\mathbf{a}$, the vector $\mathbf{c}$ and the scalar b (tensile stiffness).

We consider a straight rod and we construct a second-order theory for it. It is conenient to separate the vector components in the xy plane perpendiuclar to the rod axis: $\mathbf{u} = \mathbf{u}_\perp + u_z\mathbf{k}$, where $\mathbf{u}_\perp = u_x\mathbf{i} + u_y\mathbf{j}$, etc. In the general case $\Gamma_n = \mathbf{e}_n\cdot\mathbf{r}' - \mathbf{e}_{n0}\cdot\mathbf{r}_0'$. For a rod with a straight axis $\mathbf{r}_0' = \mathbf{k},\ \mathbf{r}' = \mathbf{k} + \mathbf{u}'$. Expressing $\mathbf{e}_n$ in conformity with (4.3) and retaining second-order terms in the equalities $\Gamma = \Gamma_3,\ \Gamma_\alpha = 0$ we obtain

$$\Gamma = u_z' - \mathbf{k}\cdot\boldsymbol{\theta}_\perp \times \mathbf{u}_\perp' - {}^1/_2\theta_\perp^2 \quad (\theta_\perp = |\boldsymbol{\theta}_\perp|)$$

$$\mathbf{u}_\perp' - \boldsymbol{\theta}_\perp \times \mathbf{k}\,(1 + u_z') + \theta_z(\mathbf{u}_\perp' \times \mathbf{k} + {}^1/_2\boldsymbol{\theta}_\perp) = 0$$

Solving the last equation for $\theta_\perp$, we arrive at the equalities

$$\theta_\perp = (1 - u_z')\mathbf{k}\times\mathbf{u}_\perp' + {}^1/_2\theta_z\mathbf{u}_\perp' \quad \Gamma = u_z' + {}^1/_2 u_\perp'^2 \tag{5.2}$$

The expression for $\varkappa$ from (4.3) can now be converted to the form

$$\varkappa = \mathbf{k}\times[(1 - u_z')\mathbf{u}_\perp']' + \theta_z'(\mathbf{k} + \mathbf{u}_\perp') + {}^1/_2\mathbf{u}_\perp'\times\mathbf{u}_\perp'' \tag{5.3}$$

We limit ourselves later to the following simple modification:

$$\mathbf{M}_0 = 0,\quad Q_0 = 0,\quad \mathbf{c} = 0,\quad \mathbf{a} = a\mathbf{e}_\alpha\mathbf{e}_\alpha + a_3\mathbf{e}_3\mathbf{e}_3$$

$$\varepsilon = 0,\quad \mathbf{J} = J\mathbf{e}_3\mathbf{e}_3$$

Compared to what is ordinarily necessary for applications, here there is just one constraint, equality of the bending stiffnesses. The elasticity relationship $\mathbf{M} = \mathbf{a}\cdot\varkappa$ is written in the form

$$M_z = a_3\theta_z' + (a - {}^1/_2\, a_3)\,\mathbf{k}\cdot\mathbf{u}_\perp' \times \mathbf{u}_\perp'' \tag{5.4}$$
$$\mathbf{M}_\perp = a\mathbf{k} \times [(1 - u_z')\,\mathbf{u}_\perp']' + a_3\theta_z'\mathbf{u}_\perp'$$

The components Λ_α in the force expression

$$\mathbf{Q} = \mathbf{Q}_\perp + Q_z\mathbf{k} = \Lambda_\alpha\mathbf{e}_\alpha + Q\mathbf{e}_3 \tag{5.5}$$

are the connection reactions for which the elasticity relationships are not written down. Using the expressions for Q and Γ from (5.1) and (5.2) and discarding third-order terms, we obtain from (5.5)

$$Q_z = b\,(u_z' + {}^1/_2 u_\perp'^2) - \mathbf{u}_\perp'\cdot\mathbf{Q}_\perp \tag{5.6}$$

The moment balance Eq.(4.4) reduces in this case to the equalities

$$M_z' + \mathbf{k}\cdot\mathbf{u}_\perp' \times \mathbf{Q}_\perp + m_z = \rho J\,(\theta_z^{\cdot\cdot} - {}^1/_2\,\mathbf{k}\cdot\mathbf{u}_\perp' \times \mathbf{u}_\perp'')$$

$$\mathbf{M}_\perp' + \mathbf{k}\,(1 + u_z') \times \mathbf{Q}_\perp + \mathbf{u}_\perp' \times Q_z\mathbf{k} + \mathbf{m}_\perp = \rho J\,(\mathbf{u}_\perp'\theta_z^{\cdot})^{\cdot}$$

From the latter it follows that

$$\mathbf{Q}_\perp = -\,a\,(1 - u_z')\,[(1 - u_z')\,\mathbf{u}_\perp']'' + \mathbf{k} \times [a_3\,(\theta_z'\mathbf{u}_\perp')' - \tag{5.7}$$
$$\rho J\,(\mathbf{u}_\perp'\theta_z^{\cdot})^{\cdot} + (1 - u_z')\,\mathbf{m}_\perp] + bu_z'\mathbf{u}_\perp'$$

Now (5.6) can be rewritten with allowable error in the form

$$Q_z = b\,(u_z' + {}^1/_2\,u_\perp'^2) + \mathbf{u}_\perp'\cdot(a\mathbf{u}_\perp''' + \mathbf{m}_\perp \times \mathbf{k}) \tag{5.8}$$

Substitution of (5.8) and (5.7) and (5.4) into the impulse balance and impulse moment equations **results** in the system

$$bu_z'' - \rho u_z^{\cdot\cdot} + q_z = -\,[{}^1/_2\,bu_\perp'^2 + \mathbf{u}_\perp'\cdot(a\mathbf{u}_\perp''' + \mathbf{m}_\perp \times \mathbf{k})]' \tag{5.9}$$

$$a\mathbf{u}_\perp^{\mathrm{IV}} + \rho\mathbf{u}_\perp^{\cdot\cdot} - \mathbf{q}_\perp = \{a\,(u_z'''\mathbf{u}_\perp' + 2u_z''\mathbf{u}_\perp'' + 2u_z'\mathbf{u}_\perp''') +$$
$$\mathbf{k} \times [a\,(\theta_z'\mathbf{u}_\perp')' - \rho J\,(\mathbf{u}_\perp'\theta_z^{\cdot})^{\cdot} + (1 - u_z')\,\mathbf{m}_\perp] + bu_z'\mathbf{u}_\perp'\}'$$

$$a_3\theta_z'' - \rho J\theta_z^{\cdot\cdot} + m_z = {}^1/_2\,\mathbf{k} \times \mathbf{u}_\perp'\cdot(a_3\mathbf{u}_\perp'' - \rho J\mathbf{u}_\perp^{\cdot\cdot})' - \mathbf{u}_\perp'\cdot\mathbf{m}_\perp$$

The right sides vanish in the linear approximation, which denotes separation of the longitudinal, bending and torsional strains. Second-order terms on the right generate a weak interaction of these kinds of strains. Eqs.(5.9) can be used to analyse non-linear waves /8/.

6. Variational formulation for the linearized equations. A small strain is examined for a stressed reference configuration. The displacement $\mathbf{u}$ and rotation θ are small quantities of the same order η while the forces and moments are represented by expressions of the type $\mathbf{Q} = \mathbf{Q}_0 + \mathbf{Q}_1$, where $\mathbf{Q}_1 = O\,(\eta)$. All the second-order terms are retained in the variational equations of the virtual work principle. Substituting the expressions for $\varkappa$ and Γ from (4.3) into (2.5), we obtain

$$\Pi = \mathbf{Q}_0\cdot[\gamma - \theta \times (\mathbf{u}' + {}^1/_2\,\mathbf{r}_0' \times \theta)] + \mathbf{M}_0\cdot(\theta' + {}^1/_2\theta' \times \theta) + \tag{6.1}$$
$${}^1/_2\,(\theta'\cdot\mathbf{a}\cdot\theta' + \gamma\cdot\mathbf{b}\cdot\gamma + 2\theta'\cdot\mathbf{c}\cdot\gamma)$$

The variation Π yields the work of the internal forces with opposite sign. The expression for the work of the moments will be unusual in the work of the external forces

$$\mathbf{M}\cdot\delta\mathbf{o} = (\mathbf{M}_0 + \mathbf{M}_1)\cdot(\delta\theta + {}^1/_2\theta \times \delta\theta + \ldots) = (\mathbf{M}_0 + \mathbf{M}_1 + \tag{6.2}$$
$${}^1/_2\mathbf{M}_0 \times \theta)\cdot\delta\theta + \ldots$$

(the representation used here for the small rotation vector results from (4.2) and recalls $\varkappa$ and ω from (4.3)).

The variational equation of statics (non-variable inertial forces are added in dynamics) is written in the form

$$\int_0^l [(\mathbf{q}_0 + \mathbf{q}_1)\cdot\delta\mathbf{u} + (\mathbf{m}_0 + \mathbf{m}_1 + {}^1/_2\,\mathbf{m}_0 \times \boldsymbol{\theta})\cdot\delta\boldsymbol{\theta} - \delta\Pi\}\,ds + \tag{6.3}$$
$$(\mathbf{Q}_0^* + \mathbf{Q}_1^*)\cdot\delta\mathbf{u}\,|_0^{\,l} + (\mathbf{M}_0^* + \mathbf{M}_1^* + {}^1/_2\,\mathbf{M}_0^* \times \boldsymbol{\theta})\cdot\delta\boldsymbol{\theta}\,|_0^{\,l} = 0$$

The loads given at the endpoints are denoted by the symbol $(\ldots)^*$.

It can be shown that the first-order terms $(\mathbf{q}_0\cdot\delta\mathbf{u}$, etc.) in (6.3) cancel one another. The remaining terms form the following variational equation:

$$\int_0^l [(\mathbf{q}_1 + \mathbf{Q}_1')\cdot\delta\mathbf{u} + (\mathbf{m}_1 + \mathbf{M}_1' + \mathbf{r}_0' \times \mathbf{Q}_1 + \mathbf{u}' \times \mathbf{Q}_0)\cdot\delta\boldsymbol{\theta}]\,ds + \tag{6.4}$$
$$(\mathbf{Q}_1^* - \mathbf{Q}_1)\cdot\delta\mathbf{u}\,|_0^{\,l} + (\mathbf{M}_1^* - \mathbf{M}_1)\cdot\delta\boldsymbol{\theta}\,|_0^{\,l} = 0$$

$\mathbf{Q}_1$ and $\mathbf{M}_1$ are expressions from (3.1).

Eqs.(6.4) and (3.1) are equivalent. Finally, the purpose of the variational formulation is not the derivation of (3.1) but the construction of approximate methods (for stability problems, say).

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